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Neutron confinement and the Aharonov–Casher effect

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Abstract
We determine the (bound) ground state of a spin-1/2 chargeless particle with anomalous magnetic moment in certain Aharonov–Casher configurations. We recast the description of the system in a supersymmetric form. Then the basic physical requirements for unbroken supersymmetry are established. We comment on the possibility of *neutron confinement* in these systems.

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Aharonov and Casher (A–C) draw attention to the existence of a quantum mechanical process [1–3], where the behaviour of an uncharged dipole is affected by the presence of an electric field. Let us imagine a quantum system consisting of an electrically charged object with axial symmetry centred around, say the z axis. The nearly point magnetic dipoles are completely polarized along the positive $\mathbf{z}$ direction. It is straightforward to note that this system can be recast in a supersymmetric form [4–6]. To study supersymmetry breaking, one solves the corresponding eigenvalue problem for the ground state of the given geometrical configuration.

Here we are concerned with another application of the A–C effect. It deals with the conditions for finding the bound states of a system with *unbroken* supersymmetry. To this end we have to assume *connectness* in the configuration space in order to be able to define a normalizable ground state. The problem turns out to have exact supersymmetry only under the fulfilment of a condition for the magnitude of the charge distribution which generates the electric field. We also discuss the possibility of *breaking* supersymmetry by examining the requirements for the existence of lower energy bound states.

To be specific, let us consider a spin-1/2 chargeless particle (i.e. a neutron) with anomalous magnetic moment κ_n. The Dirac equation can be written [7] in a covariant form ($\hbar = c = 1$) as

$$\left\{\gamma_\mu p^\mu - \frac{\eta}{2}\sigma_{\mu\nu}F^{\mu\nu} - M_n\right\}\Psi(x) = 0, \tag{1}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field tensor with $\eta = \mathrm{e}\kappa_n/M_n$. The interaction term $\frac{\eta}{2}\sigma_{\mu\nu}F^{\mu\nu}$ is the usual Pauli coupling which exhibits the correct magnetic moment of the neutron.

The Aharonov–Casher effective wave equation is obtained by making $A^0(\mathbf{r}) \neq 0$, $\mathbf{B}(\mathbf{r}) = \mathbf{0}$, with $\nabla \cdot \mathbf{E}(\mathbf{r}) = 4\pi\rho(\mathbf{r})$. Equation (1) can be recast in the form of a 'minimal interaction' in the Dirac equation:

$$\{\alpha \cdot (\mathbf{p} + \mathrm{i}\eta\beta\mathbf{E}(\mathbf{r})) + \beta M_n\}\Psi(\mathbf{r}, t) = \mathrm{i}\frac{\partial}{\partial t}\Psi(\mathbf{r}, t). \tag{2}$$

For stationary states of energy E, we write

$$\Psi_E(\mathbf{r}, t) = \Psi_E(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}Et} = \begin{pmatrix} \phi_E(\mathbf{r}) \\ \chi_E(\mathbf{r}) \end{pmatrix} \mathrm{e}^{-\mathrm{i}Et}. \tag{3}$$

Thus from (2) we find [6] that

$$\{\mathbf{p}^2 + \eta\tau_3 \otimes (2\sigma_3(\mathbf{E}(\mathbf{r}) \times \mathbf{p})_3 + \nabla \cdot \mathbf{E}(\mathbf{r})) + \eta^2\mathbf{E}^2(\mathbf{r})\}\Psi_E(\mathbf{r}) = \varepsilon\Psi_E(\mathbf{r}), \tag{4}$$

where τ_3 is a z-Pauli matrix which commutes with $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, with σ_i Pauli matrices, and $\varepsilon \equiv E^2 - M_n^2$. A $N = 1$ supersymmetry algebra can be constructed in the form

$$H_{\mathrm{SS}} = \{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q, \qquad [H_{\mathrm{SS}}, Q] = [H_{\mathrm{SS}}, Q^\dagger] = 0, \tag{5}$$

with

$$H_{\mathrm{SS}}\Psi_E(\mathbf{r}) = \frac{\varepsilon}{2M_n}\Psi_E(\mathbf{r}). \tag{6}$$

Here

$$Q \equiv \frac{1}{\sqrt{2M_n}}\tau^- \otimes \sigma \cdot (\mathbf{p} - \mathrm{i}\eta\mathbf{E}(\mathbf{r})) \tag{7}$$

is the *supersymmetric charge* and $\tau^- = \frac{1}{2}(\tau_1 - \mathrm{i}\tau_2)$, where the τ_1, τ_2 are also Pauli matrices. Thus H_{SS} is invariant under Q and $Q^\dagger$. The supersymmetry (5) also has implications for the spectral properties of the Hamiltonian H_{SS}: we note that $H_{\mathrm{SS}} = \{Q, Q^\dagger\} \geqslant 0$. That is, the Hamiltonian has only non-negative eigenvalues. Let us suppose that $|E_a\rangle$ is an eigenstate of H_{SS} with positive eigenvalue $E_a > 0$. Then it follows that

$$|E_a\rangle' \propto Q^\dagger|E_a\rangle \tag{8}$$

is also an eigenstate with the same positive eigenvalue. Relations (5) and (6) are the graded algebras of a supersymmetric system consisting of a relativistic spin-$\frac{1}{2}$ particle interacting with an external electromagnetic field.

From (6) we find that the equations for ϕ_E and χ_E are decoupled. In particular, for thermal neutrons we consider the upper components of Ψ_E which satisfy

$$\left\{\mathbf{p}^2 - \frac{4\eta E(\mathbf{r})}{r}\mathbf{S} \cdot \mathbf{L} - \eta\nabla \cdot \mathbf{E}(\mathbf{r}) + \eta^2 E^2(\mathbf{r})\right\}\phi_E(\mathbf{r}) = \varepsilon\phi_E(\mathbf{r}), \tag{9}$$

where $r = |\mathbf{r}|$, with $\mathbf{L}$ the orbital angular momentum operator. The supersymmetric generators annihilate the ground state in order to have unbroken symmetry,

$$Q\phi_{(0)}(\mathbf{r}) = 0, \qquad Q^\dagger\phi_{(0)}(\mathbf{r}) = 0, \tag{10}$$

where $\phi_{(0)}$ is the ground state of the system.

The second equation of (10) is satisfied identically since in the nonrelativistic limit the lower components $\Psi_{E=M_n}$ vanish. The first one yields

$$\sigma \cdot \{\mathbf{p} - \mathrm{i}\eta\mathbf{E}(\mathbf{r})\}\phi_{(0)}(\mathbf{r}) = 0. \tag{11}$$

Without lack of generality we can set

$$\phi_{(0)}(\mathbf{r}) = \begin{pmatrix} \phi(\mathbf{r}) \\ 0 \end{pmatrix}, \qquad \chi_{(0)}(\mathbf{r}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{12}$$

Furthermore, in a system with *axial* symmetry we also have the condition $L_3\phi_{(0)}(\mathbf{r}) = 0$, i.e. $\phi_{(0)}(\mathbf{r}) = \phi_{(0)}(r)$. Here then we are concerned with states for which $E^2 = M_n^2$ ($\varepsilon = 0$).

We begin by considering a solid sphere with the uniform charge per unit volume ρ_0 centred in the origin of the laboratory frame, so that there exists an electric field

$$\mathbf{E}(\mathbf{r}) = \begin{cases} 4\pi\rho_0\mathbf{r}/3, & 0 \leqslant r \leqslant r_0; \\ 4\pi\rho_0 r_0^3\mathbf{r}\big/3r^3, & r_0 \leqslant r < \infty, \end{cases} \tag{13}$$

where r_0 is the radius of the sphere. In this circumstance there is apparently no force on the neutrons but there exists a kind of Aharonov–Bohm effect [1–3, 7]. Nevertheless, if we allow the neutrons to penetrate the sphere, we can consider the problem of the possible bound states of the neutron in this new A–C configuration.

Then from (11) we find the first-order differential equations

$$\begin{aligned} &\left(\frac{\partial}{\partial r} - \beta r\right)\phi(r) = 0, && 0 \leqslant r \leqslant r_0; \\ &\left(\frac{\partial}{\partial r} - \beta\frac{r_0^3}{r^2}\right)\phi(r) = 0, && r_0 \leqslant r < \infty, \end{aligned} \tag{14}$$

where $\beta \equiv 4\pi\rho_0\eta/3$. Thus

$$\phi(r) = \begin{cases} A\exp(-\beta r^2/2), & 0 \leqslant r \leqslant r_0; \\ B\exp\left(-\beta r_0^3\big/r\right), & r_0 \leqslant r < \infty, \end{cases} \tag{15}$$

with A, B complex constants.

Next we demand continuity of the wavefunction and its derivative at $r = r_0$. Both conditions yield the same information:

$$\frac{A}{B} = \exp\left(-\frac{1}{2}\beta r_0^2\right). \tag{16}$$

Moreover, if $\Psi_{E=M_n}$ belongs to the Hilbert space, ϕ must be normalizable in $\mathbb{R}^3$:

$$4\pi\lim_{r\to\infty}\int_0^r |\phi(r')|^2 r'^2\,\mathrm{d}r' \to 1. \tag{17}$$

However, as $r \to \infty$ this integral diverges since $\exp\left(-\beta r_0^3\big/r\right) \to 1$. Therefore supersymmetry is broken in this case.

Next we solve the general problem (9) by separation of variables: $\phi(\mathbf{r}) = \phi(r)\mathcal{Y}_{l,j,m}(\theta,\varphi)$, where in terms of the spherical harmonics Y_{lm_l},

$$\begin{aligned} \mathcal{Y}_{l,l\pm1/2,m_j}(\theta,\varphi) = \frac{1}{\sqrt{2l+1}}&\left\{\pm\sqrt{l \pm m_j + \frac{1}{2}}Y_{lm_j-1/2}(\theta,\varphi)\begin{pmatrix}1\\0\end{pmatrix}\right. \\ &\left. + \sqrt{l \mp m_j + \frac{1}{2}}Y_{lm_j+1/2}(\theta,\varphi)\begin{pmatrix}0\\1\end{pmatrix}\right\}. \end{aligned} \tag{18}$$

The radial solutions must be normalizable in the range $0 \leqslant r < \infty$, and we also demand continuity at r_0 on the corresponding solutions. For $\psi_<(r)$ ($r \leqslant r_0$) we find

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2} + \epsilon_{\pm j} - \beta^2 r^2\right)\psi_<(r) = 0, \tag{19}$$

with $\epsilon_{\pm j} \equiv \varepsilon + \beta\,(3 \mp 2\,(j - 1/2))$. Thus

$$\psi_<(r) = C_1F_1\left(\frac{l + 3/2 - \epsilon_{\pm j}}{2}; l + 3/2 + 1; \beta r^2\right) r^{l+1}\,\mathrm{e}^{-\beta r^2/2}, \tag{20}$$

where C is a complex constant and ${}_1F_1$ is the *confluent hypergeometric function*.

For $r \geqslant r_0$ we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2} + \varepsilon \mp 2\beta r_0^3 \frac{j-1/2}{r^3} - \beta^2 \left(\frac{r_0^3}{r^2}\right)^2\right) \psi_>(r) = 0. \tag{21}$$

Equation (21) has only one kind of solution: non-normalizable scattering-like states for $\varepsilon > 0$ $\left(E^2 > M_n^2\right)$ since the potential decreases as $V \sim 1/r^2$ $(1/r^3 \sim \mathrm{d}V/\mathrm{d}r)$, and also because there is a term proportional to $1/r^4$ induced by the electric moment of the particle [1].

The second case considers a small sphere with charge q_n of small radius ϵ (in the limit $q_n \to 0$) at a distance a in front of an infinitely extending conducting wall. If r' is the distance of the point of observation from the image charge q_n', the potential of the charges becomes

$$V(x, y) = \frac{q_n}{r} - \frac{q_n'}{r'}, \qquad r, r' \geqslant \epsilon, \tag{22}$$

where

$$r^2 = (a-x)^2 + y^2, \qquad r'^2 = (a+x)^2 + y^2, \tag{23}$$

with $\widehat{\mathbf{r}} \cdot \widehat{\mathbf{y}} = 0$. Thus

$$\mathbf{E}(\mathbf{r}) = -q_n \left\{\frac{a-x}{((a-x)^2 + y^2)^{3/2}} + \frac{a+x}{((a+x)^2 + y^2)^{3/2}}\right\} \widehat{\mathbf{r}}, \qquad 0 \leqslant x < \infty. \tag{24}$$

Then from (11) we get

$$\ln \phi_{(0)}(x, y) = \eta q_n \left\{\frac{1}{\sqrt{a^2 - 2ax + x^2 + y^2}} - \frac{1}{\sqrt{a^2 + 2ax + x^2 + y^2}}\right\}. \tag{25}$$

By choosing $y = 0$, we find that

$$\phi_{(0)}(x) = \exp \eta q_n \left\{\frac{1}{\sqrt{a^2 - 2ax + x^2}} - \frac{1}{\sqrt{a^2 + 2ax + x^2}}\right\}, \tag{26}$$

which is a non-normalizable ground state function, since as $a \to \infty$, the ground state $\phi_{(0)}(x) \to 1$, for any $q_n \to 0$. Furthermore, if $a \to \epsilon$ and $q_n \to 0$ then $\phi_{(0)}(x) \to 1$. Therefore supersymmetry is also broken in this case.

In the next example we take an infinitely large uniform charge distribution with density per unit volume ρ, where a symmetric infinite plane of thickness L has been removed. This situation resembles a potential well in one-dimensional quantum mechanics. In this case we have

$$\phi_{(0)}(z) = \begin{cases} A \exp\left(-\frac{1}{2}|\alpha|(z^2 - L|z|)\right), & L/2 \leqslant |z|; \\ A \exp\left(-\frac{1}{2}|\alpha|\left(\frac{1}{4} - \frac{1}{2}L\right)\right), & |z| < L/2, \end{cases} \tag{27}$$

where A is a constant and $\alpha = -\mathrm{e}\rho\kappa_n/4M_n$. Here then $\phi_{(0)}(z)$ is normalizable, supersymmetry is unbroken and then neutron confinement is achieved.

Finally, let us examine the standard 1+2 A–C configuration [6]. The problem turns out to have exact supersymmetry only under the fulfilment of a condition for the magnitude of the charge distribution which generates the electric field.

The standard A–C configuration consists of an infinite line with uniform charge per unit length λ centred along the (divergent) z axis [1]. Next we note that if we allow the neutrons to penetrate, instead of a charged line, a thin solid infinite cylinder with a given density ρ *per unit volume,* the neutrons can be trapped under certain physical conditions. This can be done by using materials such as for instance *aluminium* [10][1] which is transparent to neutrons. To this end, we consider that the cylinder generates the electric field

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \rho\mathbf{r}/2, & 0 \leqslant r \leqslant r_0; \\ \rho r_0^2\mathbf{r}/2r^2, & r_0 \leqslant r < \infty, \end{cases} \tag{28}$$

where r_0 is the radius of the cylinder and for simplicity we have chosen $\widehat{\mathbf{r}} \cdot \widehat{\mathbf{z}} = 0$. Here $\widehat{\mathbf{r}}$ and $\widehat{\mathbf{z}}$ are unit vectors in the $\mathbf{r}$ and $\mathbf{z}$ directions respectively. The neutrons are completely polarized along the positive $\mathbf{z}$ direction. They move on a plane in the presence of $\mathbf{E}$.

Then again from (11) we find the differential equations

$$\begin{aligned} &\left(\frac{\partial}{\partial r} - \beta r\right)\phi(r) = 0, \qquad 0 \leqslant r \leqslant r_0; \\ &\left(\frac{\partial}{\partial r} - \frac{\beta r_0^2}{r}\right)\phi(r) = 0, \qquad r_0 \leqslant r < \infty, \end{aligned} \tag{29}$$

where $\beta \equiv -e\rho\kappa_n/4M_n$. Thus

$$\phi(r) = \begin{cases} A\exp\left(\frac{1}{2}\beta r^2\right), & 0 \leqslant r \leqslant r_0; \\ Br^{\beta r_0^2}, & r_0 \leqslant r < \infty, \end{cases} \tag{30}$$

with A, B complex constants.

We require continuity of the wavefunction and its derivative at $r = r_0$ yielding the boundary condition $A\exp\left(\frac{1}{2}\beta r_0^2\right) = Br_0^{\beta r_0^2}$. Additionally, if $\Psi_{E=M_n}$ belongs to the Hilbert space, ϕ must be normalizable in $\mathbb{R}^2$ and thus we must require that $\beta r_0^2 < -1$. This inequality comprises a necessary requirement on the possible values of $\lambda \equiv \rho\pi r_0^2$ if we want to keep unbroken supersymmetry. As λ depends linearly on r_0^2, one can in principle set up a configuration with the required λ [1, 6]. For instance, putting c into the expression for λ, we get $|\lambda|_{\mathrm{min}} \backsimeq 4\pi M_n c^2/|e\kappa_n| \backsimeq 4.6973 \times 10^{-3}$ (C cm^{-1}). Of course, this result is independent of the charged line diameter $2r_0$. From the above we can sketch at least two main conclusions. First, in the one-dimensional systems the electric charge distribution has to be sufficiently spread out in space in order to preserve unbroken supersymmetry. If this is the case, ϕ is normalizable and thus $\Psi^{(0)}_{E=M_n}$ constitutes a bound state of the system. Additionally, in the standard two-dimensional system, the magnitude of the electric charge distribution has to be sufficiently large ($\lambda \gtrsim 4\pi M_n c^2/|e\kappa_n|$) in order to produce a bound ground state. Secondly, in both the one- and two-dimensional systems, we are not declaring that the neutron directly physically senses a force due to the electric field generated by the charge density. From the third term on the left-hand side of (9), we state that the neutron moves towards regions where the *gradient* of the electric field increases. The second term in the same equation corresponds to the appearance of an induced electric dipole moment on the particle [1].

Note that, in the standard A–C configuration, the fulfilment of the condition $E^2 \leqslant M_n^2$ would allow *thermal neutron confinement* by an electrostatic field as a physical consequence of a *purely quantum mechanical effect*. Neutron trapping is usually obtained by means of diverse magnetic trap systems [8]. Cold neutrons are extensively used: in tests of fundamental quantum theory [9], and in applied physics [10].

To treat the general eigenvalue problem, we observe that the one stated by (9) has two kinds of solutions [6]: (a) non-normalizable scattering-like states for $\varepsilon > 0\left(E^2 > M_n^2\right)$; (b) normalizable bound states for $\varepsilon < 0\ \left(E^2 < M_n^2\right)$. The energy levels are obtained by requiring that the radial solutions and their derivatives be continuous at $r = r_0$, i.e., this is the quantization condition for the remaining energy levels.

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